FLUID-DYNAMICS.

System: It is defined as an arbitrary quantity of mass of fixed identity. Everything external to the system ~~are~~ is termed as 'surrounding'. System is separated from its surroundings by its boundaries.

We can study the motion of fluid particle/s, as they move through space. This is the system approach. [Lagrangian].

Advantage: The laws of mechanics apply directly to matter and hence to the system.

Disadvantage: The mathematical analysis becomes a bit complicated.

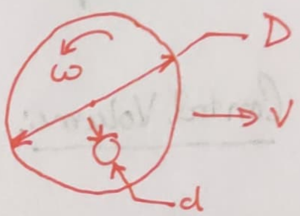
Control Volume: Alternatively, we can study a region of space as fluid flows through it. This is control volume approach (Eulerian). Example. In case of an aerofoil, we are interested in the lift and the drag. rather than what happens to the individual fluid particles. We can choose a C.V such that the aerofoil is a part of it.

Disadvantage: Since the physical laws apply to system and not directly to a region or space, we have to convert physical laws from their system formulation to a CV formulation.

Q: What are those physical laws we have been talking all along??

Quiz - 1.

P.1. Spin plays an important role in the flight trajectory of a golf ball. Therefore, it is important to know the rate at which spin decreases for a ball in flight. The aerodynamic torque, 'T'. acting on a ball in flight, is thought to depend on flight speed 'V', air density 'ρ', air viscosity, 'μ', ball diameter 'D'. ~~Spin~~ Spin rate (angular speed), ω, and diameter of the dimples on the ball, 'd'. Determine the dimensionless parameters that result.



Basic laws for a system.

1. Conservation of mass:- $\left.\frac{dM}{dt}\right|_{sys} = 0$

where, $M_{sys} = \int_{M(sys)} dm = \int_{\forall(sys)} \rho \, d\forall$.

2. Newton's Second Law:- $\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{sys}$ $\vec{P}$ = Linear momentum

where, $\vec{P}_{sys} = \int_{M(sys)} \vec{V} dm = \int_{\forall(sys)} \vec{V} \rho \, d\forall$.

3. Angular-Momentum Principle:- $\vec{T} = \left.\frac{d\vec{H}}{dt}\right)_{sys}$ $\vec{H}$ = angular momentum

$\vec{T}$ = Torque.

$\vec{H}_{sys} = \int_{M(sys)} \vec{r} \times \vec{V} dm = \int_{\forall(sys)} \vec{r} \times \vec{V} \rho \, d\forall$

4. First Law of Thermodynamics:- $\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{sys}$

$\dot{Q}$ = Heat (rate)

$\dot{W}$ = Work (rate)

$\dot{E}$ = Total energy (rate)

u = Specific Internal energy.

$E_{sys} = \int_{M(sys)} e \, dm = \int_{\forall(sys)} e \rho \, d\forall$.

$e = u + \frac{V^2}{2} + gz$.

5. Second Law of Thermodynamics:- $\left.\frac{dS}{dt}\right|_{sys} \geq \frac{1}{T} \dot{Q}$ S = entropy.

where, $S_{sys} = \int_{M(sys)} s \, dm = \int_{\forall(sys)} s \rho \, d\forall$

Next Step! To convert a system rate equation into an equivalent control volume equation.

C.V formulation.

3.3

Note:- The equation used for getting equivalent C.V formulation is known as "Reynold's Transport Theorem."

RTT is given by.

$$\left.\frac{dN}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\forall + \int_{CS} \eta \rho \vec{V} \cdot d\vec{A}$$

control volume   control surface.

where,

N :- is an extensive property

η : Intensive (per unit mass) property corresponding to N.

For,

$N = M$, $\eta = 1$

$N = \bar{P}$, $\eta = \vec{V}$

$N = \bar{H}$, $\eta = \bar{r} \times \bar{V}$

$N = E$, $\eta = e$

$N = S$, $\eta = s$

$$N_{sys} = \int_{M(sys)} \eta \, dm = \int_{\forall(sys)} \eta \rho \, d\forall$$

Physical interpretation of RTT.

$\left.\frac{dN}{dt}\right|_{sys}$ is the rate of change of the system extensive property N. if $N = M$, we obtain rate of change of mass.

$\frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\forall$ is the rate of change of the amount of N in the CV. The term $\int_{CV} \eta \rho \, d\forall$ computes the instantaneous value of N in the CV. where $\int_{CV} \rho \, d\forall$ is the instantaneous mass in the control volume.

$\int_{CS} \eta \rho \vec{V} \cdot d\vec{A}$: is the rate at which N is exiting the surface of the CV. The term $\rho \vec{V} \cdot d\bar{A}$ computes the rate of mass transfer leaving across CS area element $d\bar{A}$.

Note:- The equation used for getting equivalent C.V formulation is known as "Reynold's Transport Theorem."

RTT is given by.

$$\left.\frac{dN}{dt}\right|_{sys} = \frac{\partial}{\partial t}\underbrace{\int_{CV} \eta \rho \, d\forall}_{\text{control Volume}} + \underbrace{\int_{CS} \eta \rho \vec{V} \cdot d\vec{A}}_{\text{control surface.}}$$

where,

N :- is an extensive property

η: Intensive (per unit mass) property corresponding to N.

For,

$N = M$, $\eta = 1$

$N = \bar{P}$, $\eta = \vec{V}$

$N = \bar{H}$, $\eta = \bar{r} \times \bar{V}$

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$N = S$, $\eta = s$

$$N_{sys} = \int_{M(sys)} \eta \, dm = \int_{\forall(sys)} \eta \rho \, d\forall$$

Physical interpretation of RTT.

$\left.\frac{dN}{dt}\right|_{sys}$ is the rate of change of the system extensive property N. if $N = M$, we obtain rate of change of mass.

$\frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\forall$ is the rate of change of the amount of N in the CV. The term $\int_{CV} \eta \rho \, d\forall$ computes the instantaneous value of N in the C.V. where $\int_{CV} \rho \, d\forall$ is the instantaneous mass in the control volume.

$\int_{CS} \eta \rho \vec{V} \cdot d\vec{A}$: is the rate at which N is exiting the surface of the CV. The term $\rho \vec{V} \cdot d\bar{A}$ computes the rate of mass transfer leaving across CS area element $d\bar{A}$.

C.V formulation.

1. Conservation of Mass:

$$N = M, \quad \eta = 1$$

$$\therefore \left.\frac{dM}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} = 0$$

The first term represents the rate of change of mass within the CV

The second term represents the net rate of mass flux out through the C.S.

Note! Above equation is also known as the continuity eq.

Also, $\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = -\int_{CS} \rho \bar{V}\cdot d\bar{A}$, which implies that the rate of increase of mass in the CV is due to the net inflow of mass.

For incompressible flows,

$$\rho \frac{\partial}{\partial t}\int_{CV} d\forall + \rho \int_{CS} \bar{V}\cdot d\bar{A} = 0$$

or $$\frac{\partial \forall}{\partial t} + \int_{CS} \bar{V}\cdot d\bar{A} = 0$$

For a non-deformable CV, $\forall$ = const.

$$\therefore \int_{CS} \bar{V}\cdot d\bar{A} = 0$$

[eq. is for incompressible fluid that may be steady or unsteady]

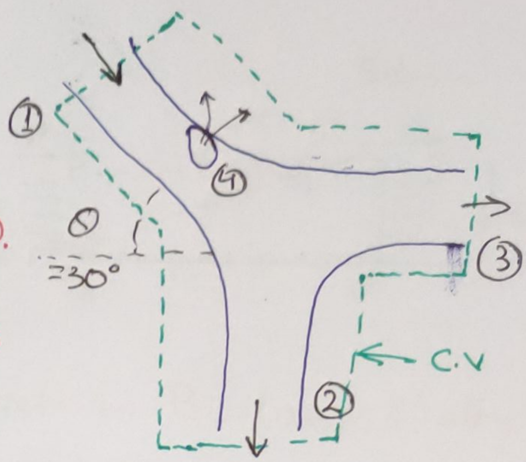
For steady, compressible flow $\partial/\partial t$ term $\rightarrow 0$.

$$\therefore \int_{CS} \rho \bar{V}\cdot d\bar{A} = 0$$

Thus for steady incompressib flow volume flow rate in and out of CV is same and for steady compressible flow mass flow rate in and out of CV is same.

Prob: Consider the steady flow in a water pipe joint. $A_1 = 0.2\ m^2$, $A_2 = 0.2 m^2$ $A_3 = 0.15 m^2$. Fluid is lost out of a hole at ④. at a rate of $0.1\ m^3/s$. $V_1 = 5 m/s$, $V_3 = 12 m/s$

Find V_2 ?



Solu. The C.V chosen is shown with dashed-line. Assuming that flow at ② is outwards also, flow is steady, incompressible with uniform properties at each section.

Governing eq:- $\int_{cs} \bar{V} \cdot d\bar{A} = 0$

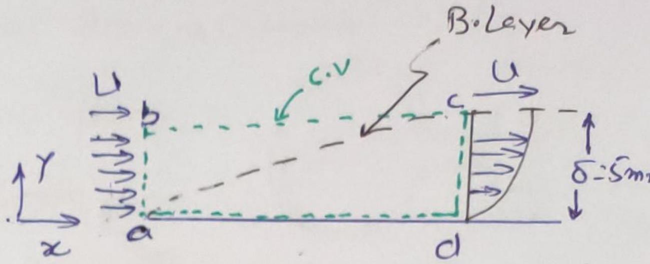
or $\bar{V}_1 \cdot \bar{A}_1 + \bar{V}_2 \cdot \bar{A}_2 + \bar{V}_3 \cdot \bar{A}_3 + Q_4 = 0$ $\qquad$ Q_4 = Leak at ④.

or $-V_1 A_1 + V_2 A_2 + V_3 A_3 + Q_4 = 0$

or $V_2 = -4.5 m/s$

The -ve sign for V_2 implies that the flow at ② is inwards.

Prob: Due to no-slip B.C. fluid in direct contact with solid boundary has zero velocity. A BL is formed as shown. The flow ahead of the plate is uniform $\vec{V} = U\hat{i}$, $U = 30\,m/s$. Velo. distr. in BL $(0 \le y \le \delta)$ along cd is given by $u/U = 2(y/\delta) - (y/\delta)^2$. Given δ at d = 5mm.



Fluid is air with $\rho = 1.24\,kg/m^3$. Assuming the plate width (w) perpendicular to the paper = 0.6m, Calculate the mass flow rate across surface bc.

Soln. Assuming steady, Incompressible, 2D flow.

$$\int_{cs} \rho \vec{V} \cdot d\vec{A} = 0$$

$$\int_{A_{ab}} \rho \vec{V} \cdot dA + \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{da}} \rho \vec{V} \cdot d\vec{A} = 0$$

no flow across da

$$\text{or } \dot{m}_{bc} = \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} = -\left[\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A}\right]$$

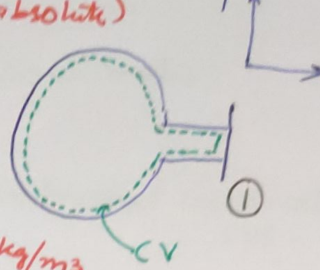
$$= -\left[-\int_{A_{ab}} \rho u\, dA + \int_{A_{cd}} \rho u\, dA\right]$$

$$= -\left[-\int_{y_a}^{y_b} \rho u w\, dy + \int_{y_a}^{y_c} \rho u w\, dy\right] = -\left[-\int_0^{\delta} \rho U w\, dy + \int_0^{\delta} \rho w U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy\right]$$

$$= -\left[-\rho U w \delta + \frac{2\rho U w \delta}{3}\right] = \frac{\rho U w \delta}{3}$$

Here +ve sign of $\dot{m}_{bc}$ indicate flow is outward at bc.

Prob: Tank of 0.05 m^3 contains air at 800 kPa (absolute) and 15°C. At $t=0$, air begins escaping from tank through a valve having area of 65 mm^2. Air passing through valve has speed of 300 m/s ; $\rho = 6\,kg/m^3$. Determine rate of change of density in the tank at $t=0$.



Soln.:

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \bar{V}\cdot d\bar{A} = 0 \quad \text{— (A)}$$

Assumption:- 1) Properties in the tank are uniform, time dependent.
2) Uniform flow at 1

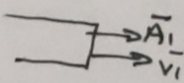
Due to uniform prop. in the tank at any instant, we can write (A) as.

$$\frac{\partial}{\partial t}\left[\rho_{CV}\int_{CV} d\forall\right] + \int_{CS} \rho \bar{V}\cdot d\bar{A} = 0$$

But $\int_{CV} d\forall = \forall$

$$\therefore \quad \frac{\partial}{\partial t}(\rho\forall) + \int_{A_1} \rho V dA = 0$$

Mass crosses CV at 1 and $\vec{V}\cdot d\bar{A}$ is +ve.



Due to uniform flow over 1,

$$\frac{\partial}{\partial t}(\rho\forall) + \rho_1 V_1 A_1 = 0$$

or $\forall \dfrac{\partial \rho}{\partial t} = -\rho_1 V_1 A_1$ ....→ $\forall$ of CV is not a fn. of time.

or $\dfrac{\partial \rho}{\partial t} = -\dfrac{\rho_1 V_1 A_1}{\forall}$

At $t=0$, $\dfrac{\partial \rho}{\partial t} = \dfrac{-6\times 300\times 65\times 10^{-6}}{0.05}$

Momentum equation.

We assume inertial C.V. i.e. the C.V coordinates are either at rest or moving at const speed w.r.t an absolute set of coordinates.

Newton's II^{nd} Law for a system moving relative to an inertial coordinate sys.

$$\vec{F} = \frac{d\bar{P}}{dt}\Big)_{sys}$$

$$\vec{P}_{sys} = \int_{M(sys)} \bar{V} dm = \int_{\forall(sys)} \bar{V} \rho d\forall$$

Also,
$$\bar{F} = \bar{F}_S + \bar{F}_B \qquad \left[\begin{array}{l} \bar{F}_S = \text{All surface forces} \\ \bar{F}_B = \text{All body forces} \end{array} \right]$$

Using Reynold's Transport Theorem.

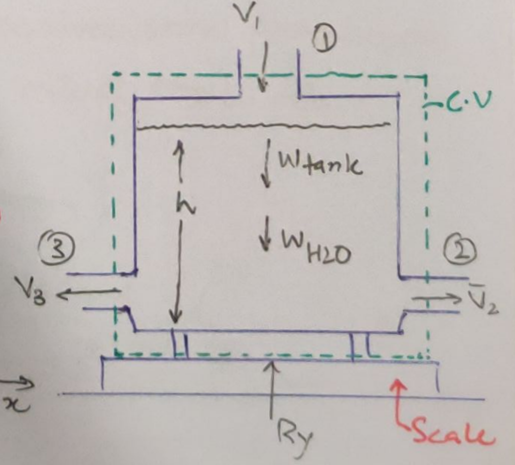
$$\frac{d\bar{P}}{dt}\Big|_{sys} = \frac{\partial}{\partial t}\int_{CV} \bar{V} \rho d\forall + \int_{CS} \bar{V} \rho \bar{V}\cdot d\bar{A}$$

or
$$\bar{F} = \bar{F}_S + \bar{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz} \rho d\forall + \int_{CS} \bar{V} \rho \bar{V}\cdot d\bar{A} \rightarrow \sum_{CS} \bar{V} \rho \bar{V}\cdot\bar{A}$$

→ for uniform flow at each inlet & exit

The total force acting on the C.V leads to a rate of change of momentum with in the C.V and/or a net rate at which momentum flux is leaving the CV through the C.S.

Prob. Metal container 2ft high, with an inside cross-sectional area of $1 ft^2$ weights 5 Lbf when empty. Container is placed on scale & water flows in through ① and out through two equal area opening ② & ③. Under steady flow condition $h = 1.9 ft$. $A_1 = 0.1 ft^2$, $\bar{V}_1 = -10\hat{j}$ ft/s, $A_2 = A_3 = 0.1 ft^2$. Your boss claims that scale will read wt. of volume of water in the tank plus the tank wt.



i.e we can treat it as a simple statics problem. You disagree, claiming that a fluid flow analysis is required. Who is right & what does the scale indicate?

Solu: R_y is the force of the scale on C.V.

<u>Assumptions</u> 1) Steady flow (2) Incompressible flow 3) Uniform flow at each inlet/outlet

<u>Governing eqs</u>: $\vec{F}_S + \vec{F}_B = \cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall} + \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A}$ — Mom. Bal.

$\frac{\partial}{\partial t}\cancelto{0}{\int_{CV} \rho\, d\forall} + \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$ — C.M

The y-comp of the momentum eq. is given by.

$$F_{Sy} + F_{By} = \int_{CS} v\rho\vec{V}\cdot d\vec{A}$$

$$R_y - W_{tank} - \underbrace{\rho g A h}_{W_{H_2O}} = \int_{A_1} v(-\rho V_1 dA_1)$$

$$= v_1(-\rho V_1 A_1)$$

[$\vec{V}\cdot d\vec{A}$ is -ve at ①
$v = 0$ at ② & ③

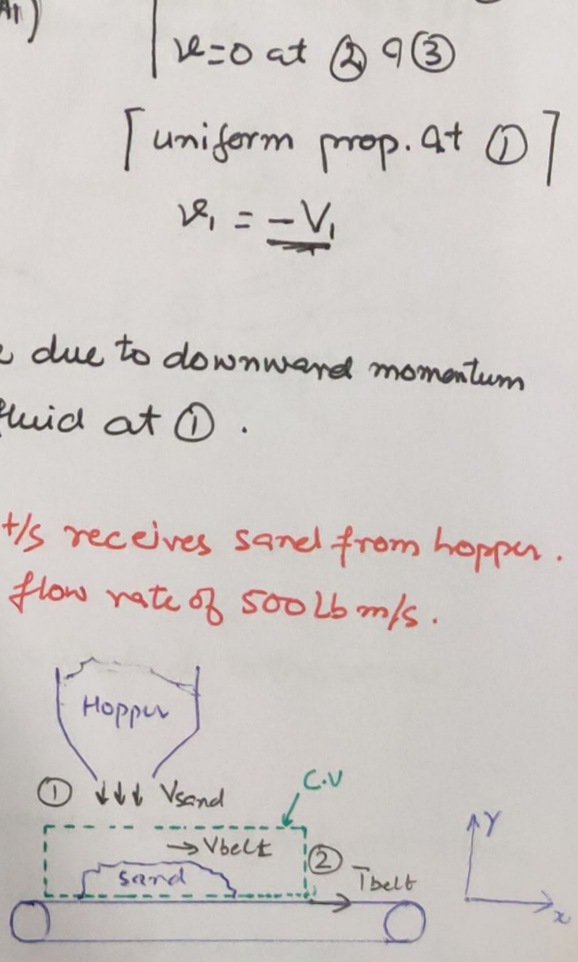
[uniform prop. at ①]

$v_1 = -V_1$

or $R_y = W_{tank} + \rho g A h + \underbrace{\rho V_1^2 A_1}$

↳ Force due to downward momentum of fluid at ①.

Prob: A horizontal conveyor belt moving at 3 ft/s receives sand from hopper. Sand falls vertically at a speed of 5 ft/s. and a flow rate of 500 lbm/s. $\rho_{sand} = 2700\ lbm/yd^3$. The conveyor belt is initially empty but begins to fill with sand. If friction in drive sys & rollers is negligible Find tension required to pull the belt while the conveyor is filling.



Assumptions!. $F_{sx} = T_{belt}$; $F_{Bx} = 0$; Uniform flow at ① ; All sand on belt moves with $V_{belt} = V_b$. Writing x-comp of momentum eq.

$$T = \frac{\partial}{\partial t}\int_{cv} u\rho d\forall + \cancel{u_1}(-\rho V_1 A_1) + \cancel{u_2}(\rho V_2 A_2)$$ [$u_1 = 0$, No flow at ②.

or $T = \frac{\partial}{\partial t}\int_{cv} u\rho d\forall$

Inside C.V, $u = V_b$ = constt.

$$\therefore T = V_b \frac{\partial}{\partial t}\int_{cv} \rho d\forall = V_b \frac{\partial M_s}{\partial t}$$

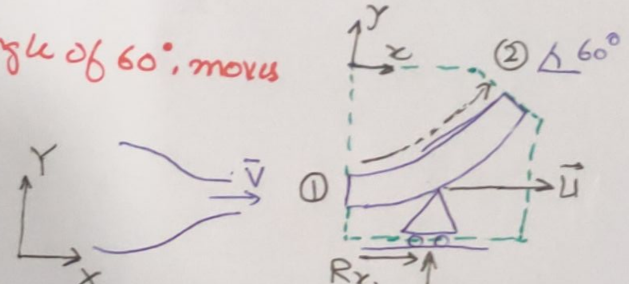
[M_s is the mass of sand on the belt inside C.V.]

Using continuity eq.

$$\frac{\partial M_s}{\partial t} = \frac{\partial}{\partial t}\int_{cv} \rho d\forall = -\int_{cs} \rho \bar{V}\cdot d\bar{A} = \dot{m}_s = 500\ lbm/s$$

$$\therefore T = V_b \dot{m}_s = 3\ ft/s \times 500\ \frac{lbm}{s} \times \frac{slugs}{32.2\ lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 46.6\ lbf$$

Problem: A vane with a turning angle of 60°, moves at a constt speed U = 10 m/s. It receives a jet of water that leaves a stationary nozzle with speed V = 30 m/s. Nozzle exit area is 0.003 m². Determine the force components that act on the vane.



Soln: A moving C.V is selected which is attached to the vane.

$\bar{U} = 10\hat{i}$, $\bar{V} = 30\hat{i}$

Assumptions!. Steady flow relative to the vane; Mag. of rel. velo along the vane $= |\bar{V}_1| = |\bar{V}_2| = V - U$.

Uniform properties at ① & ②, $F_{Bx} = 0$, Incompressible flow

Mom. eq. for C.V with rectilinear acceleration. (3.11-1)

$$\vec{F} = \bar{F}_S + \bar{F}_B = \frac{\partial}{\partial t} \int \vec{V}_{xyz}\, \rho\, d\forall + \int_{cs} \vec{V}_{xyz}\, \rho\, \vec{V}_{xyz} \cdot d\vec{A}. \quad \{\text{eq. for an inertial C.V}\}$$

In relating the system derivatives to the CV formulation $\vec{V}(x,y,z,t)$, was specified relative to the C.V coordinates x, y, z.

But the system eq. $\bar{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right|_{sys}$ where $\bar{P}_{sys} = \int_{M(sys)} \bar{V}_{XYZ}\, dm$ is valid only for velocities measured relative to an inertial ref. frame XYZ.

Since $\vec{P}_{XYZ} \neq \vec{P}_{xyz}$, when the CV ref. frame xyz is accelerating relative to the inertial ref frame, above mom. eq. is not valid for an accelerating CV.

$$\vec{V}_{XYZ} = \vec{V}_{xyz} + \vec{V}_{rf}$$

$$\therefore \frac{d\bar{V}_{XYZ}}{dt} = \bar{a}_{XYZ} = \frac{d\bar{V}_{xyz}}{dt} + \frac{d\bar{V}_{rf}}{dt} = \bar{a}_{xyz} + \bar{a}_{rf}$$

$$\text{or } \bar{F} = \int_{M(sys)} \bar{a}_{rf}\, dm + \int_{M(sys)} \frac{d\bar{V}_{xyz}}{dt}\, dm \quad \text{or } \bar{F} - \int_{M(sys)} \bar{a}_{rf}\, dm = \left.\frac{d\bar{P}_{xyz}}{dt}\right|_{sys}$$

Now, the momentum eq. for an accelerating C.V can be written as

$$\bar{F}_S + \bar{F}_B - \int_{CV} \bar{a}_{rf}\, \rho\, d\forall = \frac{\partial}{\partial t} \int_{CV} \vec{V}_{xyz}\, \rho\, d\forall + \int_{C.S} \bar{V}_{xyz}\, \rho\, \bar{V}_{xyz} \cdot d\bar{A}$$

The x-comp of momentum eq is given by.

$$F_{Sx} + \cancel{F_{Bx}}^{0} = \cancel{\frac{\partial}{\partial t}\int_{cv} u \rho \, d\forall}^{0} + \int_{cs} u \rho \vec{V}\cdot d\vec{A}$$

$$\therefore \; R_x = \int_{A1} u(-\rho V dA) + \int_{A2} u(\rho V dA) = u_1(-\rho V_1 A_1) + u_2(\rho V_2 A_2)$$

All velocities are measured relative to the coordinate axis fixed to the C.V.

Using continuity eq, we get.

$$\int_{A1}(-\rho V dA) + \int_{A2} \rho V dA = (-\rho V_1 A_1) + (\rho V_2 A_2) = 0$$

$$R_x = (u_2 - u_1)(\rho V_1 A_1)$$

$$\boxed{\begin{array}{ll} V_1 = V-U & V_2 = V-U \\ u_1 = V-U & u_2 = (V-U)\cos 60^\circ \end{array}}$$

$$R_x = [(V-U)\cos 60^\circ - (V-U)](\rho(V-U)A_1)$$

$$= -599 \text{ N}$$

[-ve sign implies dir. of R_x is opp to what we assumed]

The y-comp. of the momentum eq. is given by.

$$F_{Sy} + F_{By} = \cancel{\frac{\partial}{\partial t}\int_{cv} v \rho \, d\forall}^{0} + \int_{cs} v \rho \vec{V}\cdot d\vec{A}$$

$$R_y - Mg = \int_{A1} \cancel{v_1}^{0} \rho \vec{V}\cdot d\vec{A} + \int_{A2} v_2 \rho \vec{V}\cdot d\vec{A}$$

[$v_1 = 0$]
M = mass of C.V.

$$= v_2(\rho V_2 A_2) = v_2(\rho V_1 A_1)$$

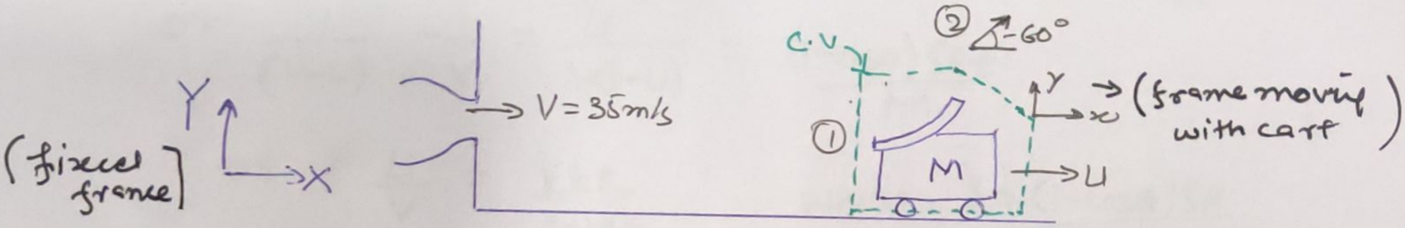
[$\rho V_1 A_1 = \rho V_2 A_2$]
$v_2 = (V-U)\sin 60^\circ$

$$= 1.04 \text{ kN}$$

$$\therefore R_y = 1.04 \text{ kN} + Mg$$

$$\vec{R} = -0.599\,\hat{i} + (1.04 + Mg)\,\hat{j} \text{ kN}$$

Prob.: A vane with $\theta = 60°$ is attached to a cart. The cart and vane of mass $M = 75$ kg, roll on a level track. Friction and air resistance may be neglected. The vane receives a jet of water which leaves a stationary nozzle horizontally at $V = 35$ m/s. The nozzle exit area $A = 0.003\,m^2$. Determine the velo. of the cart.



Assumptions: $F_{Sx} = 0$, Due to no resistance ; $F_{Bx} = 0$; mass of water in contact with vane is negligible w.r.t M. Neglect rate of change of momentum inside C.V. i.e $\frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall = 0$

Uniform flow at ① & ② ; $A_2 = A_1 = A$

Writing the x-comp. of the momentum eq.

$$\cancelto{0}{F_{Sx}} + \cancelto{0}{F_{Bx}} - \int_{CV} a_{rf_x} \rho\, d\forall = \cancelto{0}{\frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall} + \int_{CS} u \rho \vec{V}\cdot d\vec{A}$$

where, $\vec{a}_{rf}$ is the rectilinear accn. of non-inertial reference frame xyz (i.e, of the C.V) relative to inertial frame XYZ.

$$-\int_{CV} a_x \rho\, d\forall = u_1(-\rho V_1 A_1) + u_2(\rho V_2 A_2)$$

$$= (V-U)(-\rho(V-U)A) + (V-U)\cos\theta\,(\rho(V-U)A)$$

$$= -\rho(V-U)^2 A + \rho(V-U)^2 A\cos\theta$$

Also, $-\int_{CV} a_x \rho\, d\forall = -a_x M_{CV} = -a_x M = -\frac{dU}{dt} M$ =

$$= -\frac{M\,dU}{dt} = -\rho(V-U)^2 A + \rho(V-U)^2 A\cos\theta$$

(3.14)

Tut: Momentum Eq.

(1)

or $M\frac{dU}{dt} = (1-\cos\theta)\rho(V-U)^2 A$ (3.13)

or $\frac{dU}{(V-U)^2} = \frac{(1-\cos\theta)\rho A}{M}dt$

Since V = constt; $dU = -d(V-U)$. Also, $U=0$ at $t=0$; $U=U$ at $t=t$

$$\int_0^U \frac{dU}{(V-U)^2} = \int_0^U \frac{-d(V-U)}{(V-U)^2} = \frac{1}{(V-U)}\Big|_0^U = \int_0^t \frac{(1-\cos\theta)\rho A}{M}dt$$

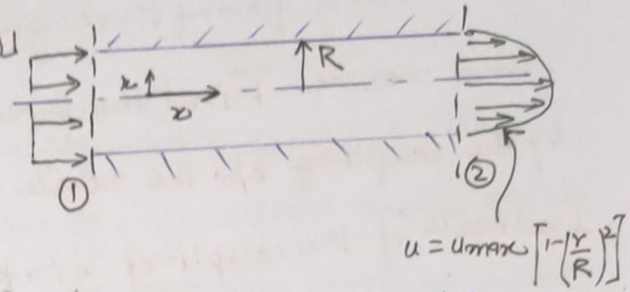
or $\frac{1}{(V-U)} - \frac{1}{V} = \frac{U}{V(V-U)} = \frac{(1-\cos\theta)\rho A t}{M}$

or $\frac{U}{V} = \frac{Vbt}{1+Vbt}$ where $b = \frac{(1-\cos\theta)\rho A}{M}$

$= \frac{0.69\,t}{1+0.69\,t}$

Tut: Momentum Eq.

Prob. Water flows through a circular pipe as shown. Evaluate the ratio of x-dir momentum flux at outlet to that at inlet.



Soln: Assumptions: Uniform flow at section ① ; Incompressible flow, Steady flow.

The x-dir momentum flux at a section is given by.

$$mf_x = \int_A u\rho V dA$$

$$\therefore \; mf_x|_1 = \int_{A_1} U\rho U dA = \rho U^2 \pi R^2$$

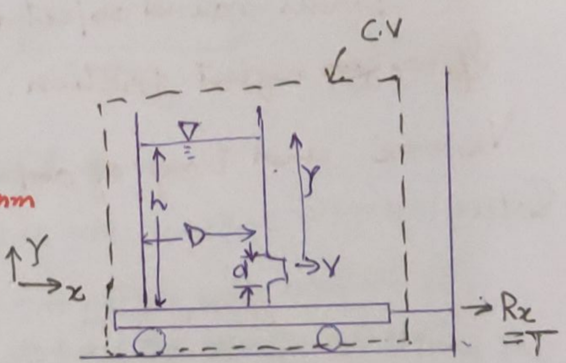
$$mf_x|_2 = \int_0^R u\rho u\, 2\pi r\, dr = 2\pi\rho u_{max}^2 \int_0^R \left[1-\left(\frac{r}{R}\right)^2\right]^2 r\,dr$$

$$= \rho u_{max}^2\, 2\pi R^2 \int_0^1 \left[1-\left(\frac{r}{R}\right)^2\right]^2 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = \rho u_{max}^2\, 2\pi R^2 \int_0^1 \left[\left(\frac{r}{R}\right) - 2\left(\frac{r}{R}\right)^3 + \left(\frac{r}{R}\right)^5\right] d\left(\frac{r}{R}\right)$$

$$= \rho u_{max}^2\, 2\pi R^2 \left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{2}\left(\frac{r}{R}\right)^4 + \frac{1}{6}\left(\frac{r}{R}\right)^6\right]_0^1 = \rho u_{max}^2 \pi R^2 \left(\frac{1}{3}\right)$$

$$\frac{mf_x|_2}{mf_x|_1} = \frac{\rho u_{max}^2 \pi R^2 (1/3)}{\rho U^2 \pi R^2} = \frac{1}{3}\left(\frac{u_{max}}{U}\right)^2$$

Prob: A large tank of height $h = 1m$ and $D = 0.6m$ is fixed to a cart. Water comes from nozzle $d = 10mm$, $V = \sqrt{2gy}$. where y is ht of free surface from nozzle. Determine tension in wire when $y = 0.8m$.



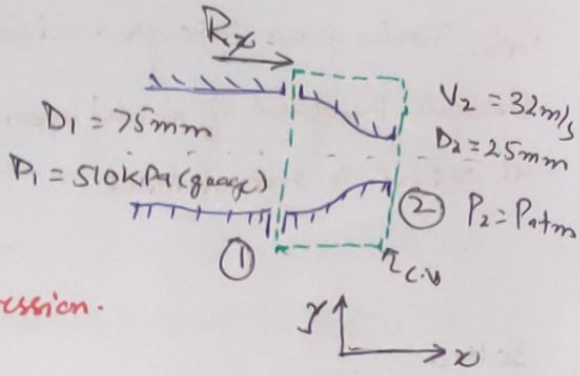
Soln: Assumption: 1) No net pr. forces 2) $F_{Bx} = 0$, 3) Steady flow (4) Uniform flow across the jet.

Writing x-comp of the momentum eq. for the inertial C.V.

$$F_{Sx} + \cancel{F_{Bx}}^{\,0} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{\,0} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$\therefore \; R_x = T = u\{\rho V_j A_{jet}\} = \rho V_j^2 A = \rho\, 2gy \frac{\pi d^2}{4}$$

Prob: Water is flowing through a fire hose and nozzle. Find the force transmitted by the coupling b/w the nozzle and hose. Indicate if the coupling is in tension or compression.



Sol: Assumptions:- 1) Steady flow 2) Uniform flow at each section 3) Incompressible flow (4) $F_{Bx} = 0$

The continuity and x-comp of mom. eq can be written as

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad \text{or} \quad 0 = -\rho V_1 A_1 + \rho V_2 A_2 \text{ or } V_1 = \frac{V_2 A_2}{A_1} = 3.56 \text{ m/s}$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho \, d\forall + \int_{CS} u \rho \vec{V}\cdot d\vec{A}$$

$$R_x + P_1 A_1 = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

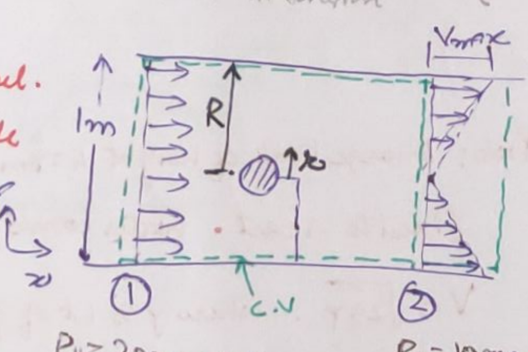
$u_1 = V_1 \quad u_2 = V_2$

[We are using guage pr. to cancel patm.]

$$\therefore R_x = -P_1 A_1 - V_1 \rho V_1 A_1 + V_2 \rho V_2 A_2 = -$$

{ Force on C.V is to left. Thus in tension }

Prob. Small round object tested in wind tunnel. If we ~~geo~~ neglect friction. Find mass flow rate $V_{2,max}$ and Drag of object.



Sol: Assumption: 1) Steady flow 2) Uniform ρ at each sec. 3) Uniform flow at (1), $\dot{m} = \rho V_1 A$. 4) $F_{Bx} = 0$

$\therefore$ Writing conser of mass & mom. balance eq,

$P_1 = 20$ mm of Hg (guage) $\quad P_2 = 10$ mm of Hg (guage)

$V_1 = 10 m/s$

$P_1 = \rho g h_1 \quad P_2 = \rho g h_2$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V}\cdot d\vec{A}$$

$\therefore \dot{m} = \rho_1 V_1 A \quad$ Also $\dot{m} = \int_{A_2} \rho_2 u_2 dA_2 = \rho_2 \int_0^R V_{2max} \frac{r}{R} 2\pi r \, dr = 2\pi \rho_2 V_{2max} R^2 \int_0^1 \left(\frac{r}{R}\right)^2 d\left(\frac{r}{R}\right)$

$$= \frac{2\pi}{3} \rho_2 V_{2max} R^2 \Rightarrow V_{2max} = \frac{3\dot{m}}{2\pi \rho_2 R^2}$$

$$R_x + P_1 A - P_2 A = u_1\{-\dot{m}\} + \int_{A_2} u \rho_2 V_2 dA_2 = -V_1 \dot{m} + 2\pi \rho_2 V_2^2 R^2 \int_0^1 \left(\frac{r}{R}\right)^3 d\left(\frac{r}{R}\right)$$

$$R_x = (P_2 - P_1)A - V_1 \dot{m} + 2\pi \rho_2 V_2^2 R^2 (1/4) = -65 N$$

R_x is the force to hold CV in place

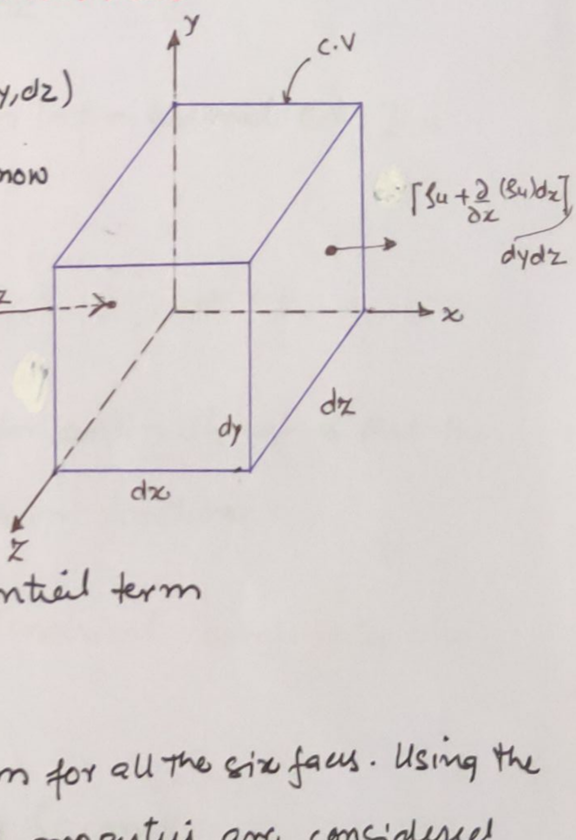
$u_1 = V_1 \quad u_2 = V_2 \frac{r}{R}$

$\Rightarrow$ Drag force $F_D = 65 N$

Differenthail approach.

Previously we developed the integral form of equations, for a C.V. They tell about the overall/gross behaviour of a flow field and its effect on various devices. However, it does not provide point by point knowledge of the flow field. In case of an aerofoil it would tell us the lift generated but not the pressure distribution that produced the lift. Differential approach provides us with details of the flow.

Differential eq. of mass conservation.



Consider an infinitesimal fixed C.V (dx, dy, dz)

Using the mass conservation relation we know from before. $\frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \bar{V}\cdot d\bar{A} = 0$

$$\int_{cv} \frac{\partial \rho}{\partial t} d\forall + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i V_i A_i)_{in} = 0$$

where $i = x, y, z$

The element is so small that the volume integral reduces to a differential term

$$\int_{cv} \frac{\partial \rho}{\partial t} d\forall \approx \frac{\partial \rho}{\partial t} dx\, dy\, dz$$

Now, we can write the mass flow term for all the six faces. Using the continuum concept, where all fluid properties are considered to be uniformly varying functions of time and position. Here we would use Taylor series expansion, such that, mass flow term is ρu at left face & $\rho u + \frac{\partial \rho u}{\partial x} dx$ at right face.

Face	Inlet mass flow	Outlet mass flow
x	$\rho u\, dy dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u) dx\right] dy dz$
y	$\rho v\, dx dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v) dy\right] dx dz$
z	$\rho w\, dx dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w) dz\right] dx dy$

Now, we could write our mass conservation relation as:

$$\frac{\partial \rho}{\partial t} dx dy dz + \frac{\partial}{\partial x}(\rho u) dx dy dz + \frac{\partial}{\partial y}(\rho v) dx dy dz + \frac{\partial}{\partial z}(\rho w) dx dy dz = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Above eq. is the cons. of mass for an infinitesimal C.V. It is also known as continuity eq.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0 \qquad \text{(Vector form)}$$

Note: The only assumption required for continuity eq. is that the density and the velocity are continuum functions.

P.S: Look for derivation in cylindrical coordinates also.

Special cases:

1) For an incompressible fluid, ρ = const.

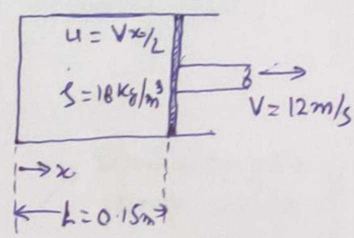
$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \vec{\nabla} \cdot \vec{V} = 0$$

2) For steady flow,

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = \vec{\nabla} \cdot \rho \vec{V} = 0$$

3.16

Problem: In the piston-cylinder system shown; at one instant when the piston is $L = 0.15m$ away from the closed end of the cylinder, the gas density is uniform at $\rho = 18\ kg/m^3$ and the piston begins to move away from the closed end at $V = 12 m/s$. Assume the gas velo. to be 1-D and proportional to the distance from the closed end; it varies linearly from zero at the end to $u = V$ at the piston. Find the rate of change of gas ρ at this instant. Obtain an expression for the average density as a fn of t.



Solu: Continuity eq. is given by,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Since $u = u(x)$, $\partial/\partial y = \partial/\partial z = 0$

$$\therefore \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\text{or } \frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} = -\rho\frac{\partial u}{\partial x} - u\cancel{\frac{\partial \rho}{\partial x}}^{\,0}$$

ρ is assumed uniform in the volume.

$$\therefore \frac{\partial \rho}{\partial t} = -\rho\frac{\partial u}{\partial x} = -\rho\frac{\partial}{\partial x}\left(\frac{Vx}{L}\right) = -\frac{\rho V}{L}$$

where $L = L_0 + Vt$

Separating variables and integrating

$$\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = -\int_0^t \frac{V}{L}dt = -\int_0^t \frac{V\,dt}{L_0 + Vt}$$

$$\text{or } \ln\frac{\rho}{\rho_0} = \ln\frac{L_0}{L_0 + Vt} \qquad \text{or } \rho = \rho_0\left[\frac{1}{1 + Vt/L_0}\right]$$

At $t = 0$,

$$\frac{\partial \rho}{\partial t} = -\frac{\rho_0 V}{L} = -\frac{18 \times 12}{0.15} = -1440\ \frac{kg}{m^3 \cdot s}$$

Euler's Equation: Momentum eq. for frictionless flow.

Euler's eq. is given by. [·

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p \quad \text{—(1)}$$

Note: Euler eq. states that for an inviscid ~~flow~~ fluid, the change in momentum of a fluid particle is caused by the body forces and the net pr. force. In reality truly inviscid fluids do not exist, but many flow problems can be successfully analysed assuming $\mu = 0$. e.g. [in aerodynamics].

x-comp of eq (1) can be written as.

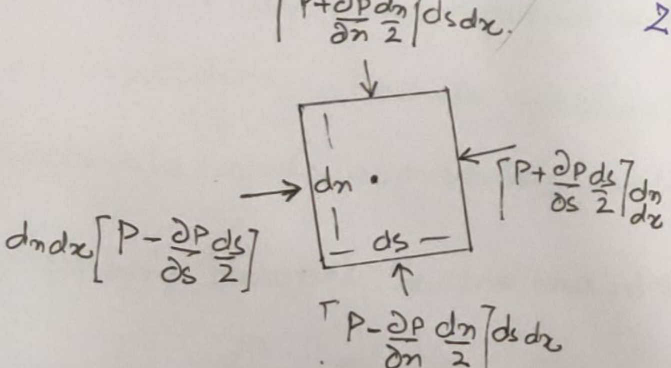
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x}$$

Similarly y & z comp. can be written.

P.S: Look at Euler's eq in cylindrical coordinates also

Note: Euler's eq. is valid when we have no viscous stresses. This could be due to $\mu = 0$, or when we have no fluid deformation. i.e rigid body motion.

Euler's eq. in Streamline coordinates:



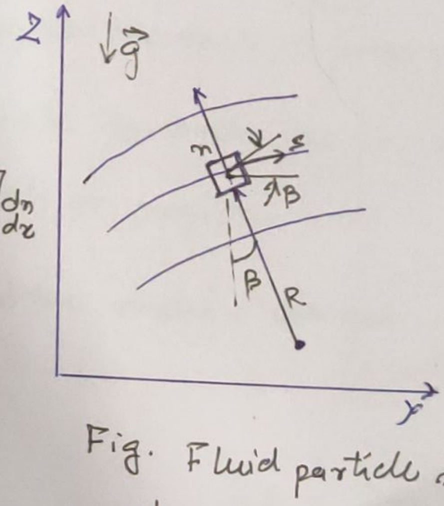


Fig. Fluid particle moving along a streamline.

Consider the flow in yz plane. Write eq. of motion along s, distance along streamline, and n, distance normal to the streamline. Applying Newton's Second law on the fluid element of volume $ds\,dn\,dx$, we get.

$$\left(P-\frac{\partial P}{\partial s}\frac{ds}{2}\right)dn\,dx-\left(P+\frac{\partial P}{\partial s}\frac{ds}{2}\right)dx\,dn-\rho g\sin\beta\, ds\,dn\,dx=\rho a_s\,ds\,dn\,dx \rightarrow 1$$

where, a_s is the accn. of the fluid particle along the streamline.

$$\text{or}\quad -\frac{\partial P}{\partial s}-\rho g\sin\beta=\rho a_s \rightarrow 2$$

$$\text{or}\quad -\frac{1}{\rho}\frac{\partial P}{\partial s}-g\frac{\partial z}{\partial s}=a_s \rightarrow 3$$

since $\sin\beta=\partial z/\partial s$

Along any streamline $V=V(s,t)$, and the total accn is given by.

$$a_s=\frac{DV}{Dt}=\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s} \rightarrow 4$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial s}-g\frac{\partial z}{\partial s}=\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s} \rightarrow 5$$

{Euler's eq. in streamwise dir, with z-axis directed vertically upwards}

For steady flow, and neglecting body forces. Euler's eq. reduces to

$$\frac{1}{\rho}\frac{\partial P}{\partial s}=-V\frac{\partial V}{\partial s} \rightarrow 6$$

Note: Above eq. indicates that for an incompressible, inviscid flow a decrease in velocity is accompanied by an increase in pressure.

Physical interpretation: Particle accelerates towards low pr. region and decelerates while approaching high pr. region.

Now, Applying Newton's Second law in the n-dir. we get.

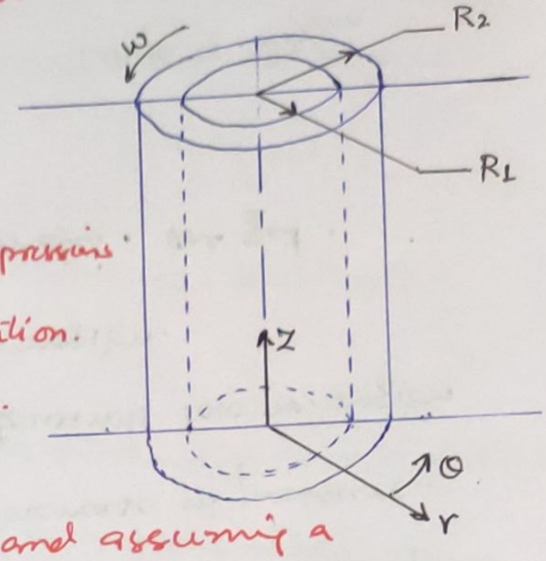
$$-\frac{\partial P}{\partial n}-\rho g\cos\beta=\rho a_n$$

$$\text{or}\quad -\frac{1}{\rho}\frac{\partial P}{\partial n}-g\frac{\partial z}{\partial n}=a_n$$

$\cos\beta=\partial z/\partial n$

a_n is the centripetal accn. $\therefore\ a_n=-V^2/R$

Prob. A viscous fluid fills the annular gap b/w vertical concentri cylinders. The inner cyl is stationary and the outer cyl. rotates at ω. The flow is laminar. Obtain the continuity, N-S, and tangential shear stress equations to model this flow field. Obtain expressions for the liquid velo. profile and the shear stress distribution. Compare the shear stress at the surface of the inner cyl. with that computed from a planar approximation. obtained by "unwrapping" the annulus into a plane and assuming a linear velo. profile across the gap. Determine the ratio of cyl radii for which the planar approx. predicts the correct shear stress at the surface of the inner cylinder within 1%.



Solu: Assumptions: 1: Steady flow 2: Incompressible flow; ρ = constt.

3). $\partial/\partial z = 0$, $V_z = 0$ 4. Circumferentially sym. flow, $\partial/\partial\theta = 0$.

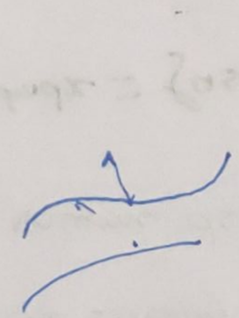
The continuity eq. is given by.

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \cancelto{0}{\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)} + \cancelto{0}{\frac{\partial}{\partial z}(v_z)} = 0$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0$$

Since, $\partial/\partial\theta = \partial/\partial z = 0$. then $\frac{\partial}{\partial r} = \frac{d}{dr}$. Integrating above eq. we get

$$r v_r = \text{constt.}$$



where R is the radius of curvature of the streamline at the chosen pt.
Now, the Euler's eq. normal to the streamline can be written as.

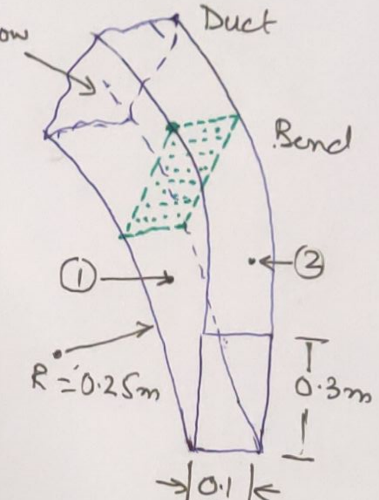
$$\frac{1}{\rho}\frac{\partial p}{\partial n} + g\frac{\partial z}{\partial n} = \frac{V^2}{R}$$

For steady flow in horizontal plane, above eq. reduces to.

$$\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}$$

Note: Above eq. indicates that pr. increases in the dir. outward from the centre of curvature of the streamlines. In regions where the streamlines are straight, $R \to \infty$. so there is no pr. variation.

Prob. Determine the flow rate of air at standard conditions by installing pr. taps across a bend. If the measured pr. difference b/w the taps is 40 mm of water, compute the approx. flow rate.



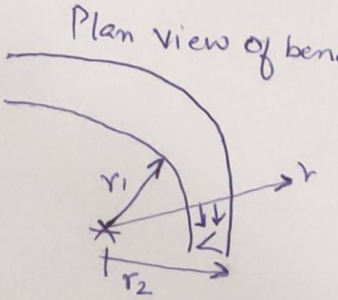
Solu:

$$P_2 - P_1 = \rho_{H_2O}\, g\, \Delta h.$$

$\Delta h = 40$ mm of H_2O.

Euler's eq. normal to streamline is given by.

$$\frac{\partial P}{\partial r} = \frac{\rho V^2}{r}$$



$P = P(r)$

$$\therefore \frac{\partial P}{\partial r} = \frac{dP}{dr} = \frac{\rho V^2}{r}$$

or $$dP = \rho V^2 \frac{dr}{r}$$

Integrating, we get

$$P_2 - P_1 = \rho V^2 \ln r \Big|_{r_1}^{r_2} = \rho V^2 \ln \frac{r_2}{r_1}$$

or $$V = \left[\frac{P_2 - P_1}{\rho \ln(r_2/r_1)}\right]^{1/2}$$

where R is the radius of curvature of the streamline at the chosen pt.
Now, the Euler's eq. normal to the streamline can be written as.

$$\frac{1}{\rho}\frac{\partial p}{\partial n} + g\frac{\partial z}{\partial n} = \frac{V^2}{R}$$

For steady flow in horizontal plane, above eq. reduces to.

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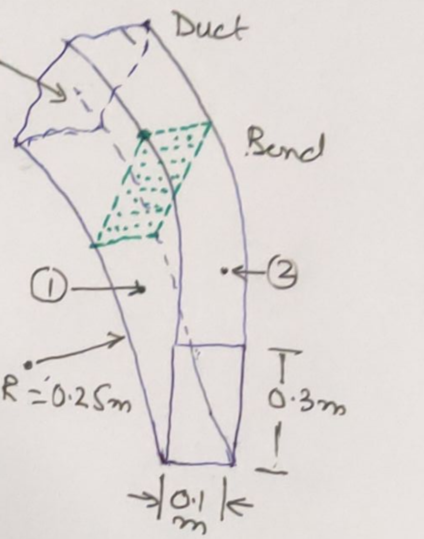
Prob. Determine the flow rate of air at standard conditions by installing pr. taps across a bend. If the measured pr. difference b/w the taps is 40 mm of water, compute the approx. flow rate.



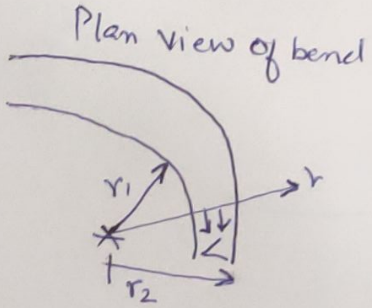
Soln:

$$P_2 - P_1 = \rho_{H_2O}\, g\, \Delta h.$$

$\Delta h = 40$ mm of H_2O.

Euler's eq. normal to streamline is given by.

$$\frac{\partial P}{\partial r} = \frac{\rho V^2}{r}$$



$P = P(r)$

$$\therefore \frac{\partial P}{\partial r} = \frac{dP}{dr} = \frac{\rho V^2}{r}$$

or $dp = \rho V^2 \frac{dr}{r}$

Integrating, we get

$$P_2 - P_1 = \rho V^2 \ln r \Big|_{r_1}^{r_2} = \rho V^2 \ln\frac{r_2}{r_1}$$

or $$V = \left[\frac{P_2 - P_1}{\rho \ln(r_2/r_1)}\right]^{1/2}$$

But $\Delta p = p_2 - p_1 = \rho_{H_2O}\, g\, \Delta h$

$$V = \left[\frac{\rho_{H_2O}\, g\, \Delta h}{\rho \ln r_2/r_1}\right]^{1/2} = \left[\frac{999 \times 9.81 \times 0.04}{1.23} \times \frac{1}{\ln(0.35/0.25)}\right]^{1/2}$$

$= 30.8$ m/s

$Q = VA = 30.8 \times 0.1 \times 0.3 = 0.924$ m^3/s

Differential Control Volume Analysis : Bernoulli eq.

Bernoulli's Equation.

Euler's eq. for a steady flow along a streamline is.

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = V\frac{\partial V}{\partial s} \quad \text{— (1)}$$

For a fluid particle, moving a distance, ds, along a streamline,

$\frac{\partial p}{\partial s} ds = dp$ (change in pr. along s)

$\frac{\partial z}{\partial s} ds = dz$ (change in elevation along s)

$\frac{\partial V}{\partial s} ds = dV$ (change in speed along s)

After multiplying eq (1) by ds, we get

$$-\frac{dp}{\rho} - g dz = V dV \quad \text{or} \quad \frac{dp}{\rho} + V dV + g dz = 0 \quad \text{(along s)}$$

After integration, we get

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad \text{—(2) (along s)}$$

For compressible flow

For applying eq. 2. to a problem, relation b/w p & ρ needs to be known. For incompressible flow, ρ = constt. Bernoulli's eq. can be written as.

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constt} \quad \text{—(3)}$$

For incompressible flow

Bernoulli's equation can only be used if the following restrictions/conditions are satisfied:

1). Steady flow 2) Incompressible flow 3) Frictionless flow
4) Flow along a streamline.

Note: 1) In aerodynamics the gravity term is usually negligible, so eq (3) indicates that wherever the velocity is relatively high (upper surface of an airfoil), the pressure must be relatively low and vice-versa.

2. In general, (if the flow is not constrained in some way), if a particle increases it elevation ($z\uparrow$) or moves into a higher pressure region ($p\uparrow$), it will tend to decelerate ($V\downarrow$)

3. Bernoulli's equation can only be applied if the restriction, discussed before apply. For eq. using Bernoulli equation we cannot explain the pressure drop in a horizontal constant diameter pipe flow. As according to Bernoulli eq. for z = constt and V = constt, $\underline{p = \text{constt}}$!!

Derivation of Bernoulli eq. using rectangular coordinates.

Euler's eq. is given by: [for steady flow].

$$\frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = (\vec{V}\cdot\nabla)\vec{V} = -\frac{1}{\rho}\nabla p - g\hat{k} \quad -(1)$$

For steady flow $V = V(x,y,z)$. For a steady flow, streamlines, path lines and streak lines coincide. The motion of a particle along streamline is governed by eq (1). During time interval dt, the particle moves $\vec{ds}$ along the streamline.

Taking dot product of eq (1) with $\vec{ds}$. we get.

$$(\vec{V}\cdot\nabla)\vec{V}\cdot d\vec{s} = -\frac{1}{\rho}\nabla p\cdot d\vec{s} - g\hat{k}\cdot d\vec{s} \quad -(2)$$

where $d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ (along s)

Now, $-\frac{1}{\rho}\nabla p\cdot d\vec{s} = -\frac{1}{\rho}\left[\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right]\cdot\left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right]$

$$= -\frac{1}{\rho}\left[\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right] \text{ (along s)}$$

$$= -\frac{1}{\rho}dp \text{ (along s)}$$

& $-g\hat{k}\cdot d\vec{s} = -g\hat{k}\cdot\left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right] = -g\,dz$ (along s)

$$(\vec{V}\cdot\vec{\nabla})\vec{V}\cdot d\vec{s} = \left[\frac{1}{2}\vec{\nabla}(\vec{V}\cdot\vec{V}) - \vec{V}\times(\vec{\nabla}\times\vec{V})\right]\cdot d\vec{s}$$ [using vector identity for $(\vec{V}\cdot\nabla)\vec{V}$]

(along s) $=0$ as $(\vec{\nabla}\times\vec{V})\perp\vec{V}$ $\Rightarrow \vec{V}\times(\vec{\nabla}\times\vec{V})\perp ds$

$$= \frac{1}{2}\left[\hat{i}\frac{\partial V^2}{\partial x} + \hat{j}\frac{\partial V^2}{\partial y} + \hat{k}\frac{\partial V^2}{\partial z}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$= \frac{1}{2}\left[\frac{\partial V^2}{\partial x}dx + \frac{\partial V^2}{\partial y}dy + \frac{\partial V^2}{\partial z}dz\right] = \frac{1}{2}d(V^2) \quad \text{(along s)}$$

Substituting these terms in eq (2) we get.

$$\frac{dp}{\rho} + \frac{1}{2}d(V^2) + g\,dz = 0 \quad \text{(along s)}$$

Integrating above eq. we get

$$\int\frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constt} \quad \text{(along s)}$$

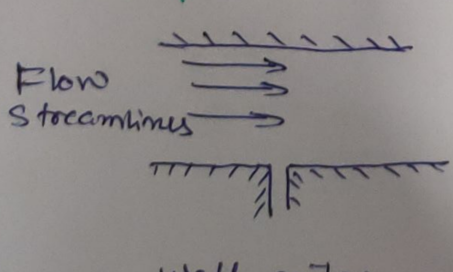
If density is constt, we obtain Bernoulli's eq.

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constt}$$

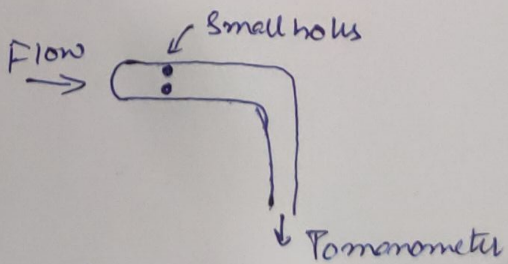
P.S: The restrictions applied to the above eq. are 1) Steady, 2) Incompressible 3) frictionless flow 4) flow along a streamline.

Static, Stagnation and Dynamic Pressure

- The pressure, p used in Bernoulli eq. is the thermodynamic pr. a.k.a the static pr. The static pr. is the pr. experienced by the fluid particle as it moves.
- There is no pr. variation normal to straight streamlines. This fact makes it possible to measure the static pr. in a flowing fluid using a wall pr. tap placed in a region where the flow streamlines are straight. The axis of the pr. tap/hole is perpendicular to the surface.



Wall pr. tap



Static pr. probe.

The stagnation pr. is obtained when a flowing fluid is decelerated to a zero speed by a frictionless process. For, incompressible flow, using Bernoulli eq., and neglecting elevation differences, we get,

$$\frac{P}{\rho} + \frac{V^2}{2} = \text{constt.}$$

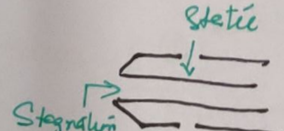
or $$\frac{P_0}{\rho} + \frac{V_0^2}{2} \nearrow 0 = \frac{P}{\rho} + \frac{V^2}{2}$$ [Subscript '0' indicates stagnation]

or $$P_0 = P + \frac{1}{2}\rho V^2$$

The term $\frac{1}{2}\rho V^2$ is called the dynamic pr.

or $$V = \sqrt{\frac{2(P_0 - P)}{\rho}}$$: V = flow speed.

For measuring stagnation pr., a probe with a hole facing directly upstream is used. This probe is called "pitot tube". Using a "pitot-static" tube, simultaneous measurements of both the static and the stagnation pr. can be performed.



Pitot-static tube

Momentum eq.

Unsteady Bernoulli Equation.

Momentum eq. for frictionless flow can be written as

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p - g\hat{k}$$

Above eq. is a vecter eq. It can be converted to a scalar eq.

$$\frac{D\vec{V}}{Dt}\cdot d\vec{s} = \frac{DV}{Dt}ds = \underbrace{V\frac{\partial V}{\partial s}ds}_{dV} + \frac{\partial V}{\partial t}ds = -\frac{1}{\rho}\underbrace{\nabla p\cdot d\vec{s}}_{dp} - g\underbrace{\hat{k}\cdot d\vec{s}}_{dz}$$

or $$VdV + \frac{\partial V}{\partial t}ds = -\frac{dp}{\rho} - gdz$$

Integrating along a streamline from point 1 to point 2, yields

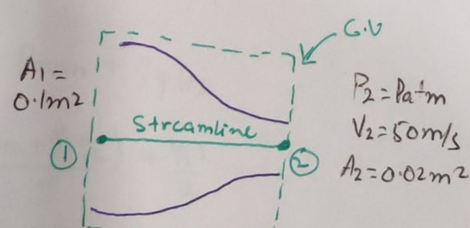
$$\int_1^2 \frac{dp}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) + \int_1^2 \frac{\partial V}{\partial t}ds = 0$$

For incompressible flow,

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t}ds$$

Restrictions! 1, Incompressible flow. 2) Frictionless flow.
3) Flow along a stream line.

Prob. Air flows steadily at low speed through a nozzle. Determine the guage pr. required at the inlet to produce $V_2 = 50$ m/s.



Solu:. Assumption:- 1, Steady flow, 2. Incompressible flow $\rightarrow M < 0.3$ for $V = 50m/s$, 3) Frictionless flow
4). Flow along a streamline 5) $z_1 = z_2$ 6) Uniform flow at ① & ②.

Gov. eq. $$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Also, $$\sum_{cs} \vec{V}\cdot\vec{A} = 0$$

$$P_1 - P_{atm} = P_1 - P_2 = \rho\frac{(V_2^2 - V_1^2)}{2}$$

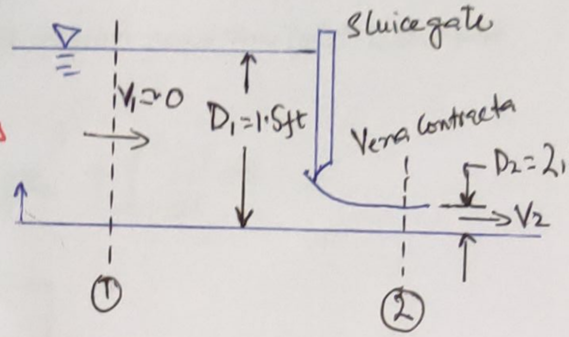
Applying continuity eq. we get.

$$(-\rho V_1 A_1) + (\rho V_2 A_2) = 0 \quad \text{or} \quad V_1 A_1 = V_2 A_2 \quad \text{or} \quad V_1 = V_2 \frac{A_2}{A_1}$$

For air at standard condition, $\rho = 1.23\ kg/m^3$

$$P_1 - P_{atm} = \frac{\rho}{2}(V_2^2 - V_1^2)$$

Problem: Water flows under a sluice gate. At the vena contracta, the flow streamlines are straight. Determine the flow speed down stream from the gate and discharge in ft^3/s per foot of width.



Solⁿ: Assumptions: 1, Steady 2, Incompressible, 3 frictionless flow 4. Uniform at each section 5. Flow along a streamline. 6, $p \propto$ depth.

= Under the above assumptions, we can apply Bernoulli eq.

⇒ Consider the streamline that runs along the bottom of the channel ($z=0$).

$$\therefore P_1 = P_{atm} + \rho g D_1 \qquad P_2 = P_{atm} + \rho g D_2$$

4. $$\frac{(P_{atm} + \rho g D_1)}{\rho} + \frac{V_1^2}{2} = \frac{(P_{atm} + \rho g D_2)}{\rho} + \frac{V_2^2}{2}$$

$$\text{or} \quad \frac{V_1^2}{2} + g D_1 = \frac{V_2^2}{2} + g D_2 \qquad \text{— (1)}$$

Now, considering the streamline that runs along the free surface on both sides.

$$\frac{P_{atm}}{\rho} + \frac{V_1^2}{2} + g D_1 = \frac{P_{atm}}{\rho} + \frac{V_2^2}{2} + g D_2$$

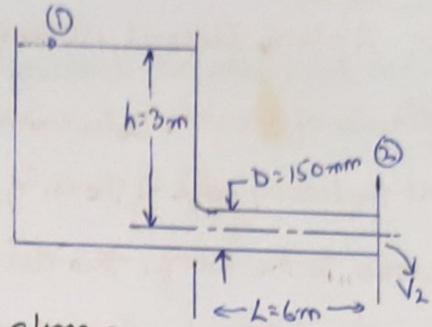
$$\text{or} \quad \frac{V_1^2}{2} + g D_1 = \frac{V_2^2}{2} + g D_2 \qquad \text{— (2)}$$

We see that eq (1) = eq (2),

$$\therefore V_2 = \sqrt{2g(D_1 - D_2) + V_1^2} \qquad \text{But } V_1 \simeq 0$$

$$= \sqrt{2g(D_1 - D_2)} \rightarrow Q = VA = VDw \ \therefore Q/w = VD = V_2 D_2$$

Prob. A long pipe is connected to a large reservoir that initially is filled with water ($h = 3m$). Determine the flow velocity leaving the pipe as a function of time after a cap is removed from its free end.



Solu: 1, Incompressible 2, Frictionless flow. 3, Flow along a streamline from ① to ②. 4, $P_1 = P_2 = P_{atm}$ 5, $V_1^2 \simeq 0$ 6, $z_2 = 0$, 7, $z_1 = h = $ constt.
8, Neglect velocity in reservoir, except for small region near the inlet to the tube.

Then,

Governing eq:
$$\cancel{\frac{P_1}{\rho}} + \cancel{\frac{V_1^2}{2}} + gz_1 = \cancel{\frac{P_2}{\rho}} + \frac{V_2^2}{2} + \cancel{gz_2} + \int_1^2 \frac{\partial V}{\partial t}\, ds$$

Then,
$$gz_1 = gh = \frac{V_2^2}{2} + \int_1^2 \frac{\partial V}{\partial t}\, ds$$

In view of assumption (8)
$$\int_1^2 \frac{\partial V}{\partial t}\, ds \approx \int_0^L \frac{\partial V}{\partial t}\, ds$$

$$\int_0^L \frac{\partial V}{\partial t}\, ds = \int_0^L \frac{dV_2}{\partial t}\, ds = L\frac{dV_2}{dt}$$

(In the tube everywhere $V = V_2$)

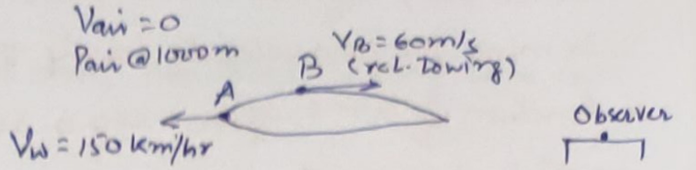
$$\frac{dV_2}{2gh - V_2^2} = \frac{dt}{2L}$$

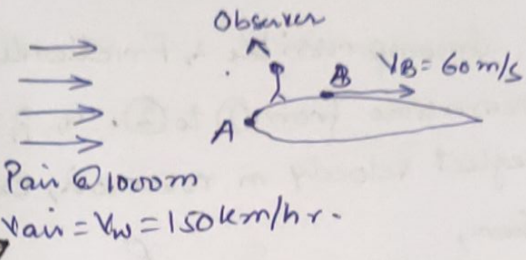
Using Limits, $V = 0$, at $t = 0$ and $V = V_2$ at $t = t$

$$\int_0^{V_2} \frac{dV}{2gh - V^2} = \left[\frac{1}{\sqrt{2gh}} \tanh^{-1}\left(\frac{V}{\sqrt{2gh}}\right)\right]_0^{V_2} = \frac{t}{2L}$$

or $\frac{1}{\sqrt{2gh}} \tanh^{-1}\left(\frac{V_2}{\sqrt{2gh}}\right) = \frac{t}{2L}$ or $\frac{V_2}{\sqrt{2gh}} = \tanh\left(\frac{t}{2L}\sqrt{2gh}\right)$

Prob: A plane flies at 150 km/hr at an altitude of 1000m. Determine the stagnation pr. at the leading edge of the wing. At a certain pt. close to the wing, the air speed relative to the wing is 60 m/s. Compute the pr. at this pt.





Solu: Flow is unsteady when observed from a fixed frame. However, an observer on the wing sees the steady flow

At $Z = 1000$ m in std air, $T = 281$ K. and speed of sound is 336 m/s. Hence at B, $M_B = V_B/c = 0.178$. which is less than 0.3, so flow is incompressible.

Assumptions: 1) Steady, 2) Incompressible 3) frictionless flow, 4) Neglect ΔZ, 5) Flow along a streamline.

Gov. eq: $\frac{P_{air}}{\rho} + \frac{V_{air}^2}{2} + gz_{air} = \frac{P_A}{\rho} + \frac{V_A^2}{2} + gz_A = \frac{P_B}{\rho} + \frac{V_B^2}{2} + gz_B$

At, 1000 m, $P/P_{SL} = 0.8870$, $\rho/\rho_{SL} = 0.9075$,.

$\therefore$ $P = 0.8870 P_{SL} = 0.8870 \times 1.01 \times 10^5 = 8.96 \times 10^4 \ N/m^2$

$\rho = 0.9075 \rho_{SL} = 0.9075 \times 1.23 = 1.12 \ kg/m^3$

Since, $V_A = 0$ at the stagnation pt.

$P_{0A} = P_{air} + \frac{1}{2}\rho V_{air}^2 = 8.96 \times 10^4 + \frac{1}{2} \times 1.12 \left(\frac{150 \times 1000}{3600}\right)$

$P_B = P_{air} + \frac{1}{2}\rho(V_{air}^2 - V_B^2)$

$\left(\frac{150 \times 1000}{3600}\right)^2 - (60)^2$

P.S: 1) The Bernoulli equation Interpreted as energy equation → Derivation, Problems

2) Velocity Potential function, How it is related to stream function.

Momentum eq.

For an infinitesimal system of mass dm, Newton's Second Law can be written as,

$$d\vec{F} = dm \left.\frac{d\vec{V}}{dt}\right)_{sys.}$$

$$= dm \frac{D\vec{V}}{Dt} = dm\left[u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}\right]$$

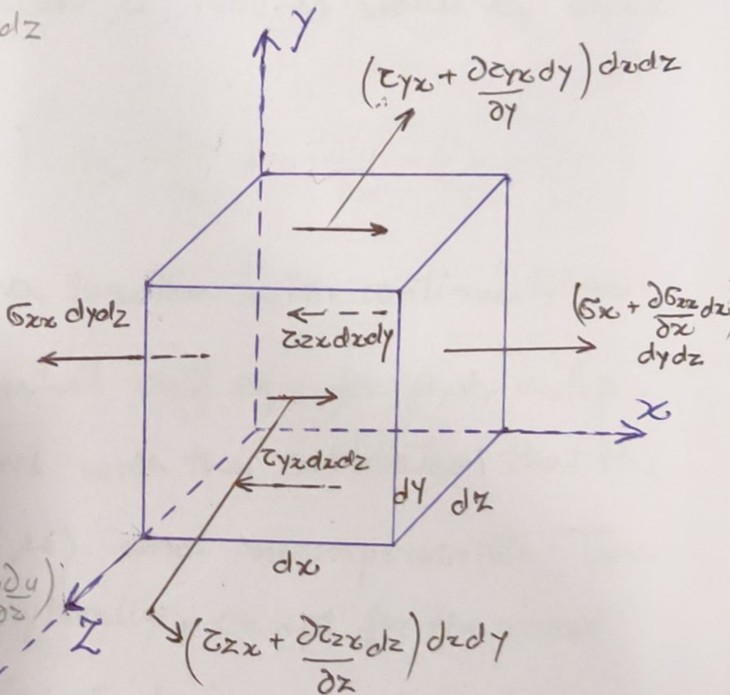
Now, we need to obtain an expression for $d\vec{F}$. For that, we consider the x comp. of the force acting on a differential element of mass dm and volume $d\forall = dx\,dy\,dz$. The forces acting on $d\forall$ can be both ~~Su~~ body forces and surface forces. Surface forces include both normal & shear forces.

$$dF_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx\,dy\,dz$$

Considering body force $dF_{Bx} = \rho g_x$

$$dF_x = dF_{sx} + dF_{Bx}$$

$$= \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx\,dy\,dz$$



∴ Eq. of motion:- x - comp.

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Note: Write y & z comp. [Home assignment]

Above is the diff eq. of motion for any fluid satisfying the continuum approximation.

Note: Before we can solve above eq. we need to obtain expressions for the stress terms in terms of velo. & pr.

For a Newtonian-fluid the viscous stress $\propto$ the rate of shear strain. (angular deformation rate). ∴

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) ; \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) ; \quad \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{xx} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial u}{\partial x} ; \quad \sigma_{yy} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial v}{\partial y} ; \quad \sigma_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial w}{\partial z}$$

where, p is the local thermodynamic pr. p is related to temp. & density by equation of state.

$$\rho\frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

Above eq. of motion is called Navier-Stokes equation. For an incompressible flow with constant μ. Navier-Stokes eq. can be greatly simplified.

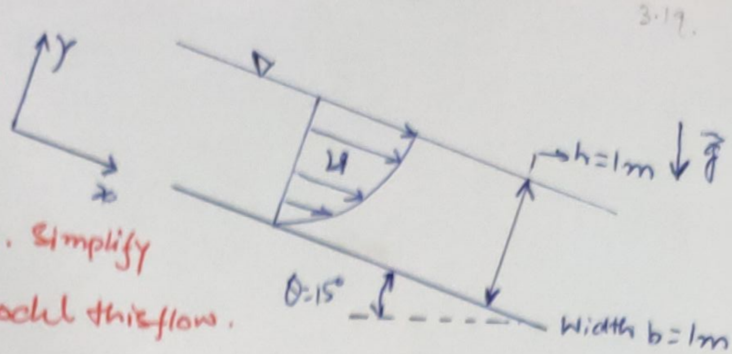
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

The above eq along with y & z comp. together with continuity eq. form 4 eq. coupled nonlinear partial diff eq. for u, v, w, & p. These eq. describe many flows with the restriction that the fluid be Newtonian (i.e const μ). and incompressible. These eq. are impossible to solve analytically, except for the most basic cases. CFD is used for studying more realistic flows. For the case of frictionless flow ($\mu = 0$) the eq. of motion reduce to Euler's equation.

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Prob. 3.19.

A liq flows down an inclined plane surface in a steady, fully developed laminar film of thickness, h. Simplify the continuity & N-S equations to model this flow. Obtain expressions for the liquid velocity profile, the shear stress distribution, the vol. flow rate, and the avg. velo. Relate the liq. film thickness to the vol. flow rate per unit depth of the surface normal to the flow. Calculate the vol. flow rate in a film of water h = 1mm thick, flowing on a surface b = 1m wide, $\theta = 15°$.



Solu:- Assumptions. 1. Steady flow, 2. Incomp. flow; ρ = constt.

3. No flow or properties variation in z-dir; $w = 0$, $\partial/\partial z = 0$.

4. Fully developed flow; $\partial/\partial x = 0$.

Now the governing eq. can be written as,

$$\cancelto{0}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancelto{0}{\frac{\partial w}{\partial z}} = 0$$

$$\rho\left(\cancelto{0}{\frac{\partial u}{\partial t}} + \cancelto{0}{u\frac{\partial u}{\partial x}} + \cancelto{0}{v\frac{\partial u}{\partial y}} + \cancelto{0}{w\frac{\partial u}{\partial z}}\right) = \rho g_x - \cancelto{0}{\frac{\partial p}{\partial x}} + \mu\left(\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{0}{\frac{\partial^2 u}{\partial z^2}}\right)$$

$\rightarrow \frac{\partial v}{\partial y} = 0$, Also, from assumption 3 & 4, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial z} = 0$, $\Rightarrow$ v is constt.

Since v is zero at solid surface, then, v must be zero everywhere.

$$\rho\left(\cancelto{0}{\frac{\partial v}{\partial t}} + \cancelto{0}{u\frac{\partial v}{\partial x}} + \cancelto{0}{v\frac{\partial v}{\partial y}} + \cancelto{0}{w\frac{\partial v}{\partial z}}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\rho\left(\cancelto{0}{\frac{\partial w}{\partial t}} + \cancelto{0}{u\frac{\partial w}{\partial x}} + \cancelto{0}{v\frac{\partial w}{\partial y}} + \cancelto{0}{w\frac{\partial w}{\partial z}}\right) = \cancelto{0}{\rho g_z} - \frac{\partial p}{\partial z} + \mu\left(\cancelto{0}{\frac{\partial^2 w}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 w}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 w}{\partial z^2}}\right)$$

After simplification.

$$0 = \rho g_x + \mu\frac{\partial^2 u}{\partial y^2}$$

$$0 = \rho g_y - \frac{\partial p}{\partial y}$$

Since $\frac{\partial u}{\partial z} = 0$, and $\frac{\partial u}{\partial x} = 0$. $\Rightarrow$ u is at most a fnc of y.

$$\therefore \quad \frac{d^2u}{dy^2} = -\frac{\rho g_x}{\mu} = -\rho g \frac{\sin\alpha}{\mu} \qquad \left[\frac{\partial^2 u}{\partial y^2} = \frac{d^2u}{dy^2}\right]$$

Integrating,

$$\frac{du}{dy} = -\rho g \frac{\sin\alpha}{\mu} y + c_1$$

& $$u = -\rho g \frac{\sin\alpha}{\mu} \frac{y^2}{2} + c_1 y + c_2.$$

The boundary conditions, B.C. are. $u=0$ @ $y=0$. [no-slip]
Also, $du/dy=0$, at $y=h$, [zero shear-stress at liq. free surface].
Evaluating, we get

$$0 = 0 + 0 + c_2 \implies c_2 = 0$$

& $$0 = -\rho g \frac{\sin\alpha}{\mu} h + c_1 \quad \text{or} \quad c_1 = \rho g \frac{\sin\alpha}{\mu} h$$

Substituting c_1 & c_2 we get,

$$u = -\rho g \frac{\sin\alpha}{\mu} \frac{y^2}{2} + \rho g \frac{\sin\alpha}{\mu} hy = \rho g \frac{\sin\alpha}{\mu}\left(hy - \frac{y^2}{2}\right)$$

Shear stress, $\tau_{yx} = \mu \frac{du}{dy} = \rho g \sin\alpha (h-y)$.

$$Q = \int_A u\,dA = \int_0^h u\,b\,dy \qquad \text{[where } b = \text{width in z-dir]}$$

$$= \int_0^h \rho g \frac{\sin\alpha}{\mu}\left(hy - \frac{y^2}{2}\right) b\,dy = \rho g \frac{\sin\alpha\, b}{\mu}\left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h$$

$$= \rho g \frac{\sin\alpha}{\mu} b \frac{h^3}{3}$$

Avg. flow velo. $= Q/A = Q/bh = \rho g \frac{\sin\alpha}{\mu} \frac{h^2}{3}$

Film thickness, $h = \left[\frac{3\mu Q}{\rho g \sin\theta\, b}\right]^{1/3}$